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A Bound for the Modulus of Continuity for Metric Projections in a Uniformly Convex and Uniformly Smooth Banach Space

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In 1979, Bjornestal obtained a local estimate for a modulus of uniform continuity of the metric projection operator on a closed subspace in a uniformly convex and uniformly smooth Banach space B. In the present paper we give the global version of this result for the projection operator on an arbitrary closed convex set in B.

1. INTRODUCTION AND PRELIMINARIES

Metric projection operators P_Ω on convex closed sets Ω (in the sense of best approximation) are widely used in theoretical and applied areas of mathematics, expecially connected with problems of optimization and approximation. As examples one can consider iterative-projection methods for solving equations, variational inequalities and minimization of functionals [1], and methods of alternating projections for finding common points of convex closed sets in Hilbert spaces [10, 8, 9].

Let us recall the definition of the metric projection operator. Let B be a real uniformly convex and uniformly smooth (reflexive) Banach space with B^* its dual space, Ω a closed convex set in B, and $\langle w, v \rangle$ a dual product in B, i.e., a pairing between $w \in B^*$ and $v \in B$ ($\langle y, x \rangle$ is an inner product in Hilbert space H, if we identify H and H^*). The signs $\|\cdot\|$ and $\|\cdot\|_{B^*}$ denote the norms in the Banach spaces B and B^*, respectively.

DEFINITION 1.1. The operator $P_\Omega : B \to \Omega \subset B$ is called a metric projection operator if it yields a correspondence between an arbitrary point $x \in B$ and its nearest point $\bar{x} \in \Omega$ according to the minimization problem

$$P_\Omega x = \bar{x}; \qquad \bar{x}: \|x - \bar{x}\| = \lim_{\xi \in \Omega} \|x - \xi\|. \tag{1.1}$$

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Metric projection operators have extremely good properties in Hilbert spaces [14, 20, 1]. However this is not generally true in Banach spaces. For example, operators P_Ω do not posses such important properties as monotonicity, non-expansiveness and absolutely best approximation [1], which make the metric projection operators in Hilbert spaces exceptionally effective. To illustrate this let us recall the properties of a metric projection operator on a subspace M of a Hilbert space. Here, such an operator is orthogonal, linear, non-expansive, self-adjoint and idempotent [10]. Metric projection operators on a subspace M of a Banach space have no such properties in general [13].

However, P_Ω does possess a number of good qualities realized in very important applications [2, 3]. For example, it is uniformly continuous in a Banach space B on each bounded set and satifies the basic variational principle [16] (see also [11, 18])

$$\langle J(x-\bar{x}), \bar{x}-\xi \rangle \geqslant 0, \qquad \forall \xi \in \Omega. \tag{1.2}$$

Here $J\colon B \to B^*$ is duality mapping in B defined by the equalities [16, 1]

$$\langle Jx, x \rangle = \|Jx\|_{B^*} \|x\| = \|x\|^2.$$

The smoothness properties of the metric projection operator have been studied for a long time. In Hilbert space it satisfies the Lipschitz condition and, consequently, it is uniformly continuous. It is known that in a uniformly convex Banach space the metric projection operator is always continuous but not always uniformly continuous.

The results of F. Murray and J. Lindenstrauss (see [14]) suggest the following problem: "Is the operator P_Ω uniformly continuous in a uniformly convex and uniformly smooth Banach space?" In 1979, B. Bjornestal obtained a positive answer to this question in the form of the estimate [7]

$$\|P_M x - P_M y\| \leqslant 2\delta_B^{-1}(2\rho_B(6\|x-y\|)), \tag{1.3}$$

where M is a closed linear subspace of B, $\rho_B(\tau)$ is a modulus of smoothness, $\delta_B(\varepsilon)$ is a modulus of convexity of the space B, and $\delta_B^{-1}(\cdot)$ is the inverse function to $\delta_B(\varepsilon)$ [15]. *But this result was only local* (it is fulfilled if x and y are sufficiently near to each other and $\|x-\bar{x}\|=1$, $\|y-\bar{y}\|=1$).

Recently, in the paper [19] the following global estimate was established in a uniformly convex and uniformly smooth Banach space B

$$\|P_\Omega x - P_\Omega y\| \leqslant \|x-y\| + 4C_1\,\delta_B^{-1}(N\psi(\|x-y\|/C_1)) \tag{1.4}$$

where N is some fixed constant, $C_1 = \|x - P_\Omega y\| \vee \|P_\Omega x - y\|$, and ψ is the function defined by the formula

$$\psi(t) = \int_0^t \frac{\rho_B(s)}{s}\, ds.$$

In [4] we obtained another estimate of the uniform continuty of the metric projection operator in a uniformly convex and uniformly smooth Banach space B:

$$\|P_\Omega x - P_\Omega y\| \leqslant C g_B^{-1}(N C g_{B^*}^{-1}(N \|x - y\|)), \tag{1.5}$$

where $g_B(\varepsilon) = \delta_B(\varepsilon)/\varepsilon$, $g_B^{-1}(\cdot)$ is an inverse function, $N = 2LC$, L is constant, $1 < L < 3.18$, (see [12]) and

$$C = 2 \max\{1, \|x - P_\Omega y\|, \|y - P_\Omega x\|\}.$$

However, simple calculations show that Bjornestal's estimate (1.3) is better than (1.5), firstly by comparing their orders. For instance, known estimates for the moduli of convexity and smoothness of the space l^p, L^p and W^p_m, where $\infty > p > 1$,

$$\rho_B(\tau) \leqslant p^{-1}\tau^p, \qquad \delta_B(\varepsilon) \geqslant (p-1)\,\varepsilon^2/8, \qquad 1 < p \leqslant 2,$$

$$\rho_B(\tau) \leqslant (p-1)\,\tau^2, \qquad \delta_B(\varepsilon) \geqslant p^{-1}(\varepsilon/2)^p, \qquad \infty > p > 2$$

give the following orders: for (1.3)

$$\|P_\Omega x - P_\Omega y\| \sim \|x - y\|^{2/p}, \qquad p \geqslant 2,$$

and for (1.5)

$$\|P_\Omega x - P_\Omega y\| \sim \|x - y\|^{1/(p-1)}, \qquad p \geqslant 2$$

Note that for $p > 2$ we have $2/p > 1/(p-1)$.

Let now $1 < p \leqslant 2$. Then the estimate (1.3) yields

$$\|P_\Omega x - P_\Omega y\| \sim \|x - y\|^{p/2}, \qquad 2 \geqslant p > 1,$$

and for estimate (1.5) we have

$$\|P_\Omega x - P_\Omega y\| \sim \|x - y\|^{p-1}, \qquad 2 \geqslant p > 1.$$

Note again that for $1 < p < 2$ we have $p/2 > p - 1$. For $p = 2$ (Hilbert case) (1.3) and (1.5) give the same orders:

$$\|P_\Omega x - P_\Omega y\| \sim \|x - y\|.$$

Let us emphasize again that (1.3) is a local estimate. In the next section we will obtain its global form for arbitrary closed convex set Ω in a Banach space.

2. Auxiliary Theorems

The lower and upper parallelogram inequaliteis and estimates of duality mappings in uniformly convex and uniformly smooth Banach spaces (respectively) obtained first in [5, 6, 17] are used as the basis in order to prove uniform continuity of te metric projection operators in Banac spaces. In tis section we will prove two auxiliary theorems.

THEOREM 2.1. *In uniformly smooth Banach space B the following estimate*

$$\langle Jx - Jy, x - y \rangle \leqslant 8 \, \|x - y\|^2 + C\rho_B(\|x - y\|), \qquad \forall x, y \in B \quad (2.1)$$

is valid, where

$$C = C(\|x\|, \|y\|) = 4 \max\{2L, \|x\| + \|y\|\}.$$

Proof. Denote

$$D = 2^{-1}(\|x\|^2 + \|y\|^2 - \|2^{-1}(x + y)\|^2/2)$$

and consider two possibilities:

(i) Let $\|x + y\| \leqslant \|x - y\|$. Then

$$\|x\| + \|y\| \leqslant \|x + y\| + \|x - y\| \leqslant 2 \, \|x - y\|.$$

Squaring this expression, we obtain

$$2^{-1} \, \|x\|^2 + 2^{-1} \, \|y\|^2 + \|x\| \, \|y\| \leqslant 2 \, \|x - y\|^2.$$

Now, let us subtract $\|2^{-1}(x + y)\|^2$ from both sides of this inequality. We have

$$D \leqslant 2 \, \|x - y\|^2 - (\|2^{-1}(x + y)\|^2 + \|x\| \, \|y\|).$$

If $\|2^{-1}(x + y)\|^2 + \|x\| \, \|y\| \geqslant \|x - y\|^2$ then immediately

$$D \leqslant \|x - y\|^2. \quad (2.2)$$

Suppose that the opposite inequality occurs. In this case, it is easily verified that

$$2^{-1}\,\|x\|^2+2^{-1}\,\|y\|^2-\|2^{-1}(x+y)\|^2$$
$$\leqslant \|2^{-1}(x+y)\|^2+\|x\|\,\|y\|\leqslant \|x-y\|^2$$

which follows from the estimate $(\|x\|-\|y\|)^2\leqslant \|x+y\|^2$, i.e., (2.2) is valid.

(ii) Let now $\|x+y\|\geqslant \|x-y\|$. It can be shown that

$$\|x\|+\|y\|-\|x+y\|\leqslant \varepsilon(x, y) \tag{2.3}$$

where

$$\varepsilon(x, y)=\|x+y\|\,\rho_B\left(\frac{\|x-y\|}{\|x+y\|}\right). \tag{2.4}$$

Indeed, let us replace

$$x=\tfrac{1}{2}(u+v), \qquad y=\tfrac{1}{2}(u-v)$$

and set

$$\alpha=\frac{u}{\|u\|}, \qquad \beta=\frac{v}{\|u\|}.$$

Using the definition of the modulus of smoothness $\rho_B(\tau)$, one can write

$$\begin{aligned}\|x\|-\|y\|-\|x+y\| &= 2^{-1}(\|u+v\|+\|u-v\|)-\|u\|\\ &= 2^{-1}\,\|u\|\,(\|\alpha+\beta\|+\|\alpha+\beta\|-2)\\ &\leqslant \|u\|\,\sup[2^{-1}(\|\alpha+\beta\|+\|\alpha-\beta\|)\\ &\qquad -1\mid \|\alpha\|=1,\ \|\beta\|=\tau]\\ &\leqslant \|u\|\,\rho_B(\|\beta\|).\end{aligned}$$

Returning to the previous notation we obtain (2.3) and (2.4). Thus,

$$\left\|\frac{x+y}{2}\right\|\geqslant\frac{\|x\|+\|y\|-\varepsilon(x, y)}{2}.$$

The right hand part is nonnegative. In fact, using the property $\rho_B(\tau)\leqslant\tau$ [15] we establish the inequality

$$\|x\|+\|y\|-\varepsilon(x, y)\geqslant \|x\|+\|y\|-\|x-y\|\geqslant 0.$$

Then

$$\left\|\frac{x+y}{2}\right\|^2 \geqslant \left(\frac{\|x\|+\|y\|}{2}\right)^2 - \varepsilon(x, y)\,\frac{\|x\|+\|y\|}{2}.$$

By virtue of $\|x\|-\|y\| \leqslant \|x-y\|$ we have

$$D \leqslant \left(\frac{\|x\|-\|y\|}{2}\right)^2 + \varepsilon(x, y)\,\frac{\|x\|+\|y\|}{2}$$

$$\leqslant \left\|\frac{x-y}{2}\right\|^2 + \varepsilon(x, y)\,\frac{\|x\|+\|y\|}{2}. \tag{2.5}$$

(a) Suppose that $\|x+y\| \leqslant 1$ then $\|x+y\|^{-1}\,\|x-y\| > \|x-y\|$. It is known [12] that the inequality

$$\tau_1^2 \rho_B(\tau_2) \leqslant L\tau_2^2 \rho_B(\tau_1), \qquad 0 \leqslant \tau_1 \leqslant \tau_2, \qquad 1 < L < 3.18 \tag{2.6}$$

holds in an arbitrary Banach space. By (2.5) and (2.6)

$$\rho_B(\|x-y\|/\|x+y\|) \leqslant L\,\|x+y\|^{-2}\,\rho_B(\|x-y\|).$$

It follows from the last estimate that

$$D \leqslant 4^{-1}\,\|x-y\|^2 + 2^{-1}L(\|x\|+\|y\|)\,\|x+y\|^{-1}\,\rho_B(\|x-y\|).$$

So for $\|x+y\| \geqslant \|x-y\|$ we have

$$2^{-1}\,\|x+y\|^{-1}\,(\|x\|+\|y\|) \leqslant (2\,\|x+y\|)^{-1}\,(\|x+y\|+\|x-y\|) \leqslant 1.$$

Therefore

$$D \leqslant 4^{-1}\,\|x-y\|^2 + L\rho_B(\|x-y\|). \tag{2.7}$$

(b) Let us now assume $\|x+y\| \geqslant 1$. Then we obtain in addition to (2.7), the form

$$D \leqslant 4^{-1}\,\|x-y\|^2 + 2^{-1}(\|x\|+\|y\|)\,\rho_B(\|x-y\|). \tag{2.8}$$

Here we used (2.5) and the convexity of $\rho_B(\tau)$. The estimates (2.2), (2.7) and (2.9) joined together give

$$2\,\|x\|^2 + 2\,\|y\|^2 + \|x+y\|^2$$
$$\leqslant 4\,\|x-y\|^2 + 2\max\{2L, \|x\|+\|y\|\}\,\rho_B(\|x-y\|).$$

This is *the upper parallelogram inequality* in a uniformly smooth Banach space [6].

Denote the right hand part of this inequality by $k(\|x-y\|)$. Then

$$D \leqslant k(\|x-y\|)/4. \tag{2.9}$$

Further, for the convex function $\phi(x)=\|x\|^2/2$, let us construct the concave (with respect to λ) function

$$\Phi(\lambda)=\lambda\phi(x)+(1-\lambda)\,\phi(y)-\phi(y+\lambda(x-y)), \qquad 0\leqslant\lambda\leqslant 1.$$

It is obvious that $\Phi(0)=0$. Suppose $0<\lambda_1\leqslant\lambda_2$. Then $\lambda_1^{-1}\Phi(\lambda_1)\geqslant \lambda_2^{-1}\Phi(\lambda_2)$, i.e., $(\Phi(\lambda)/\lambda)'\leqslant 0$. From this expression we have $\Phi'(\lambda)\leqslant\Phi(\lambda)/\lambda$. In particular, $\Phi'(1/4)\leqslant 4\Phi(1/4)$. But

$$\Phi(\tfrac{1}{4})=\tfrac{1}{4}\,\phi(x)+\tfrac{3}{4}\,\phi(y)-\phi(\tfrac{1}{4}\,x+\tfrac{3}{4}\,y).$$

It follows from (2.9) that for all z_1 and z_2

$$\phi\left(\frac{z_1+z_2}{2}\right)\geqslant\frac{\phi(z_1)}{2}+\frac{\phi(z_2)}{2}-k(\|x-y\|)/8.$$

Let us set $z_1=(x+y)/2$ and $z_2=y$ and use the property $k(t/2)\leqslant k(t)/2$. We obtain

$$\begin{aligned}\phi\left(\frac{1}{4}\,x+\frac{3}{4}\,y\right)&=\phi\left(\frac{1}{2}\left(\frac{1}{2}\,x+\frac{1}{2}\,y\right)+\frac{1}{2}\,y\right)\geqslant\phi\,\frac{1}{2}\left(\frac{x+y}{2}\right)+\frac{1}{2}\,\phi(y)-\frac{1}{8}\,k\left(\left\|\frac{x-y}{2}\right\|\right)\\&\geqslant\frac{1}{4}\,\phi(x)+\frac{1}{4}\,\phi(y)-\frac{1}{16}\,k(\|x-y\|)+\frac{1}{2}\,\phi(y)-\frac{1}{8}\,k\left(\left\|\frac{x-y}{2}\right\|\right)\\&=\frac{1}{4}\,\phi(x)+\frac{3}{4}\,\phi(y)-\frac{1}{8}\,k(\|x-y\|).\end{aligned}$$

Thus, $\Phi(1/4)\leqslant k(\|x-y\|)/8$ and

$$\Phi'(1/4)=\phi(x)-\phi(y)-\langle\phi'(y+\tfrac{1}{4}\,(x-y)),\,x-y\rangle\leqslant k(\|x-y\|)/2.$$

Furthermore, writing this inequality with y and x in place of x and y we get

$$\phi(y)-\phi(x)-\langle\phi'(x+\tfrac{1}{4}\,(y-x)),\,y-x\rangle\leqslant k(\|x-y\|)/2.$$

Summing the two last inequalities gives

$$\langle\phi'(x+\tfrac{1}{4}\,(x-y))-\phi'(y-\tfrac{1}{4}\,(x-y)),\,x-y\rangle\leqslant k(\|x-y\|).$$

One can now make a non-degenerate substitution of the variables x and y

$$z_1=2x-\tfrac{1}{2}\,(x-y), \qquad z_2=2y+\tfrac{1}{2}\,(x-y)$$

which leads to relations

$$z_1 - z_2 = x - y \qquad \text{and} \qquad \|x\| + \|y\| \leqslant \|z_1\| + \|z_2\|.$$

Taking into consideration the fact that $Jx = \phi'(x)$ is a homogeneous operator, we find

$$\langle Jz_1 - Jz_2, z_1 - z_2 \rangle \leqslant 2k(\|z_1 - z_2\|).$$

The theorem is completely proved.

The proof of the next inequality is shorter than the previous, but it has the constant L and the function $C(\|x\|, \|y\|)$ under the sign of the modulus of smoothness $\rho_B(\tau)$.

THEOREM 2.2. *In a uniformly smooth Banach space B the estimate*

$$\langle Jx - Jy, x - y \rangle \leqslant (2L)^{-1} \rho_B(8CL\|x - y\|), \qquad \forall x, y \in B \tag{2.10}$$

is valid, where

$$C = C(\|x\|, \|y\|) = 2 \max\{1, \sqrt{(\|x\|^2 + \|y\|^2)/2}\}$$

Proof. Lemma 2.1 from [4] (cf. Theorem 2 from [5]) gives the following estimate

$$\langle Jx - Jy, x - y \rangle \geqslant (2L)^{-1} \delta_{B^*}(\|Jx - Jy\|_{B^*}/C) \tag{2.11}$$

for a uniformly smooth space B. From (2.11) we have

$$\|Jx - Jy\|_{B^*} \|x - y\| \geqslant (2L)^{-1} \delta_{B^*}(\|Jx - Jy\|_{B^*}/C).$$

Since $g_{B^*}(\varepsilon) = \delta_{B^*}(\varepsilon)/\varepsilon$, we can write

$$g_{B^*}(\|Jx - Jy\|_{B^*}/C) \leqslant 2CL\|x - y\|. \tag{2.12}$$

It is known from the geometry of Banach spaces [15] that

$$\rho_B(\tau) \geqslant \varepsilon\tau/2 - \delta_{B^*}(\varepsilon), \qquad 0 \leqslant \varepsilon \leqslant 2, \qquad \tau \geqslant 0.$$

Therefore

$$\rho_B(4\delta_{B^*}(\varepsilon)/\varepsilon) \geqslant \delta_{B^*}(\varepsilon).$$

We denote $h_B(\tau) = \rho_B(\tau)/\tau$. Then

$$h_B(4g_{B^*}(\varepsilon)) \geqslant \varepsilon/4.$$

Setting

$$\varepsilon = \|Jx - Jy\|_{B^*}/C,$$

and using the non-decreasing property of the function $h_B(\tau)$, we find from (2.12)

$$h_B(4g_{B^*}(\varepsilon)) \leqslant h_B(8CL\,\|x-y\|).$$

Therefore

$$\|Jx - Jy\|_{B^*} \leqslant 4Ch_B(8CL\,\|x-y\|). \tag{2.13}$$

Now, (2.10) can be obtained from the inequality of Cauchy–Schwarz. The theorem is proved.

Remark 2.3. $C(\|x\|, \|y\|)$ in estimates (2.1) and (2.10) are absolute constants $C = 8\max\{L, R\}$ and $C = 2\max\{1, R\}$, respectively if $\|x\| \leqslant R$ and $\|y\| \leqslant R$. In these cases, (2.1) and (2.10) are a quantitative description of the property of uniform continuity (in the form of a dual product) for the duality mapping J. At the same time (2.13) gives the modulus of uniform continuity of J in traditional form.

3. Main Theorems

In this section we will provide the estimates for the conitnuity of the metric projection operator on a convex closed set of a uniformly convex and uniformly smooth Banach space B. In the case when $\|x\|$ and $\|y\|$ are uniformly bounded, they are the estimates of the moduli of a uniform continuity of this operator on each bounded set of B.

Theorem 3.1. *In a uniformly convex and uniformly smooth Banach space B the following estimate*

$$\|P_\Omega x - P_\Omega y\| \leqslant C\delta_B^{-1}(2LC_1\rho_B(\|x-y\|)), \tag{3.1}$$

is satisfied where

$$C = 2\max\{1, \|x-\bar{y}\|, \|y-\bar{x}\|\},$$

$$C_1 = 16 + 24\max\{L, \|x-\bar{y}\|, \|y-\bar{x}\|\}.$$

Proof. It is known [5] that

$$\rho_B(\tau) \geqslant \rho_H(\tau) = \sqrt{1+\tau^2} - 1 \geqslant \tau^2/(\tau+2).$$

Then

$$\rho_B(\|x-y\|) \geqslant \|x-y\|^2/(\|x-y\|+2).$$

From this we have

$$\|x-y\|^2 \leqslant (\|x\|+\|y\|+2)\, \rho_B(\|x-y\|),$$

and taking (2.1) in consideration, we obtain

$$\langle Jx-Jy, x-y\rangle \leqslant \bar{C}_1 \rho_B(\|x-y\|)$$

where

$$\bar{C}_1 = 16+24 \max\{L, \|x\|, \|y\|\}.$$

Now let us turn to estimating the convex functional $\varphi(x) = \|x\|^2$. We have

$$\|x-\bar{y}\|^2 - \|y-\bar{y}\|^2 \leqslant 2 < J(y-\bar{y}), x-y\rangle + \langle J(x-\bar{y}) - J(y-\bar{y}), x-y\rangle$$
$$\leqslant 2\langle J(y-\bar{y}), x-y\rangle + C_2 \rho_B(\|x-y\|),$$
$$C_2 = 16+24 \max\{L, \|x-\bar{y}\|, \|y-\bar{y}\|\}.$$

By analogy with the above, we can write

$$\|y-\bar{x}\|^2 - \|x-\bar{x}\|^2 \leqslant 2\langle J(x-\bar{x}), y-x\rangle + C_3\, \rho_B(\|x-y\|),$$
$$C_3 = 16+24 \max\{2L, \|x-\bar{x}\|, \|y-\bar{x}\|\}.$$

Add the last two inequalities. Then

$$(\|x-\bar{y}\|^2 - \|x-\bar{x}\|^2) + (\|y-\bar{x}\|^2 - \|y-\bar{y}\|^2)$$
$$\leqslant 2\langle J(y-\bar{y}) - J(x-\bar{x}), x-y\rangle + 2C_4 \rho_B(\|x-y\|),$$
$$C_4 = 16+24 \max\{L, \|x-\bar{y}\|, \|y-\bar{x}\|\}.$$

It is known [16] that

$$\langle J(x-\bar{x}) - J(y-\bar{y}), x-y\rangle \geqslant 0.$$

Therefore

$$(\|x-\bar{y}\|^2 - \|x-\bar{x}\|)^2 + (\|y-\bar{x}\|^2 - \|y-\bar{y}\|^2) \leqslant 2C_4 \rho_B(\|x-y\|). \quad (3.2)$$

The conditions of uniform convexity of the functional $\varphi(x) = \|x\|^2$ and uniform convexity of the Banach space B gives

$$(\|x-\bar{y}\|^2 - \|x-\bar{x}\|^2) \geqslant 2\langle J(x-\bar{x}), \bar{x}-\bar{y}\rangle + (2L)^{-1}\, \delta_B(\|\bar{x}-\bar{y}\|/C_5),$$
$$C_5 = 2 \max\{1, \|x-\bar{y}\|, \|x-\bar{x}\|\}$$

and

$$(\|y-\bar{x}\|^2 - \|y-\bar{y}\|^2) \geqslant 2\langle J(y-\bar{y}), \bar{x}-\bar{y}\rangle + (2L)^{-1}\, \delta_B(\|\bar{x}-\bar{y}\|/C_6),$$
$$C_6 = 2\max\{1, \|y-\bar{x}\|, \|y-\bar{y}\|\}.$$

Using

$$\langle J(x-\bar{x}) - J(y-\bar{y}), \bar{x}-\bar{y}\rangle \geqslant 0,$$

[16], we obtain

$$(\|x-\bar{y}\|^2 - \|x-\bar{x}\|^2) + (\|y-\bar{x}\|^2 - \|y-\bar{y}\|^2) \geqslant L^{-1}\delta_B(\|\bar{x}-\bar{y}\|/C_7),$$
$$c_7 = 2\max\{1, \|x-\bar{y}\|, y-\bar{x}\|\}. \tag{3.3}$$

It follows from (3 2) and (3.3) that

$$L^{-1}\delta_B(\bar{x}-\bar{y}\|/C_7) \leqslant 2C_4\,\rho_B(\|x-y\|).$$

Finally, we have (3.1). The theorem is proved.

Next we formulate a statement corresponding to the estimate of the duality mapping (2.10).

THEOREM 3.2. *In a uniformly convex and uniformly smooth Banach space B the following estimate*

$$\|P_\Omega x - P_\Omega y\| \leqslant C\delta_B^{-1}(\rho_B(8LC\,\|x-y\|)) \tag{3.4}$$

is satisfied, where

$$C = 2\max\{1, \|x-\bar{y}\|, \|y-\bar{x}\|\}.$$

We omit the proof of this Theorem because it is similar to the proof of the previous Theorem 3.1.

Remark 3.3. For the Hilbert space H one can write (3.4) in the form

$$\|P_\Omega x - P_\Omega y\| \leqslant 16LC^2\,\|x-y\|,$$

because $\delta_B^{-1}(\cdot)$ and $\rho_B(\cdot)$ are increasing functions, $\rho_H(\tau) \leqslant \tau^2/2$ and

$$\varepsilon^2/8 \leqslant \delta_H(\varepsilon) \leqslant \varepsilon^2/4.$$

Remark 3.4. It follows from (3.1) and (3.4) that the projection operator P_Ω is uniformly continuous on every bounded set of the uniformly convex and uniformly smooth Banach space B (cf. [19]).

It is interesting to compare the Bjornestal's estimate and a local version of our estimate (3.4). By virtue of the small distance between x and y in (1.3) and the condition $\|x-\bar{x}\|=1$ and $\|y-\bar{y}\|=1$, the constant C in (3.4) can be bounded by 2. Thus

$$\|P_\Omega x - P_\Omega y\| \leqslant 2\delta_B^{-1}(\rho_B(8LC\,\|x-y\|)), \qquad 1<L<3.18. \tag{3.5}$$

The constant in the parentheses of (3.5) is larger than the one in (1.3). This is natural because (3.4) is a global estimate. In addition, the constants in (1.3) have been obtained for the case $\Omega=M$, were M is a linear subspace of B. (One might note that the constants in the estimates (1.3), (3.1), (3.4)) and (3.5) as a rule do not have important meaning. On the contrary, the orders of estimates play the main role and they are the same there).

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